Nonequilibrium self-gravitational system

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(Dated: October 14, 2009)

The density of states of self-gravitational system diverges when the particles are spread to infinity. Other problem based an inhomogeneous distribution of particles,which motivate the gravitational interaction. In this sense the statistical mechanics of self-gravitational system is essentially an non-equilibrium problem. A new possible approach to statistical description of self-gravitational system has been proposed. The approach based on non-equilibrium statistical operator, which allow take into account inhomogeneous distribution particle and temperature in self gravitational system. The saddle point procedure, which used in the given method describes the spatially inhomogeneous distribution in self gravitational system accompanied by temperature changing.

PACS numbers: 05.70.Jk, 51.30.+i, 82.60.Lf

The statistical description of a system of interacting particles is a difficult problem but has the permanent attention for the problems of astrophysics [1].The self-gravitational system are interesting for testing ideas about the statistical mechanic description of systems governed by long range interaction. The statistical mechanics of self gravitational system turns out to be very different from that of other, more familiar, many-body systems. A self-gravitational system has also more general problem, studied for a long time [2]. The standard methods of statistical mechanic cannot be carrier to study gravitational system. Due to this fundamental difference, the notion of equilibrium is not always well defined and those system exhibit a nontrivial behaviour with gravitational collapse. For system with long-range interaction the thermodynamically ensemble are inequivalent, negative specific heat [3] in microcanonical ensemble which not exist in canonical description [4].In self-gravitational system can increase entropy and equilibrium states are only local entropy maximum. However, if introduce a repulsive potential at short distance, complete core collapse is prevented and can to proved that a global entropy maximum now exist for all accessible values of energy [7]. For the gas with pure gravitational interaction between its particles the partition function diverges, the energy of self-gravitational system are not as extensive parameter. In it was shown that a self-gravitating gas collapses. A nature of the collapse and its conditions are explained by using simple and clear consideration [8]. The phase transition in such systems creates the problem of description in the mean-field thermodynamics approach [4]. Two type approaches (statistical and thermodynamic) have been develop to determination the equilibrium states of self-gravitational system [4], [5]. About all this problem and possible solution very good described in review [6]. The collapse in such system are begin as spatially homogeneous distribution of particles in the all system at once. Formation of the spatially inhomogeneous distribution of interaction particles is a typical problem in condensed matter physics and requires non-conventional methods of statistical description of the system was tailored to gravitational interacting particles with regard for an arbitrary spatially inhomogeneous particle distribution. Spatially inhomogeneous particle distribution can not describe in standard approach equilibrium statistical physics. The microcanonical ensemble is the correct description of an isolated Hamiltonian self gravitational system at fixed energy, the canonical ensemble is the correct description of a dissipative Brownian system at fixed temperature [6]. For inhomogeneous distribution of particle can not fixed any energy or temperature. Temperature can change together with distribution of particle in particular if fixed energy of system. Very important question are in point can be considered inhomogeneous distribution of particle as equilibrium state. This state is not stationary and derivation of density produce flux of particle though changing of gravitational force which act on every particle. The statistical description of self-gravitational system must take into account the possible inhomogeneous distribution the temperature together with distribution of particles. A few model systems with interaction are known for which the partition function can be exactly evaluated, at least within thermodynamic limits [9] but not for inhomogeneous distribution of particle [10]. Inhomogeneous distribution of particle and temperature need to use nonequilibrium approach to description of behaviour self gravitational system. In this article present the possible approach local equilibrium statistical operator [14] which leads to take into account the inhomogeneous distribution particle and temperature. Formation of the spatially inhomogeneous distribution of interaction particles requires a nonconventional method, such as use in [11], [12], [13], which are based an additional field representation of statistical sum [15]. For progress the goal can use saddle point approximation which provided to nonlinear equation. Solution of this equation present real distribution particle and temperature under certain conditions. In this approach do not use the polytrophic dependence pressure from density. This dependence are result as consequence of the thermodynamic relation. Presented condition give the possibility determine statistical operator for self gravitational system and fully describe the thermodynamic behaviour of self-gravitational system.

Phenomenological thermodynamic based on the conservation lows for average value of physical parameter as number of particles, energy and impulse. Statistical thermodynamic nonequilibrium system based too on conservation lows not the average value dynamic variables but in particular for this dynamic variables. For determination thermodynamic function of nonequilibrium system are need use the presentation of corresponding statistical ensembles which take into account the nonequilibrium states of this systems. The conception of Gibbs ensembles can brings to description nonequilibrium stationary states of system. In this case can determine nonequilibrium ensemble as totality of system which be contained in same stationary external action. This system have same character of contact with thermostat and possess all possible value macroscopical parameters which compatibility present conditions. In system, which are in same stationary external condition will be formed local equilibrium stationary distribution. For exactly determination local equilibrium ensemble must accordingly determine the distribution function or statistical operator of system [14].

If assume that nonequilibrium states of system can determine through inhomogeneous distribution energy $H(\mathbf{r})$ and number of particles (density $n(\mathbf{r})=\sum_i \delta(\mathbf{r}-\mathbf{r_i})$) the local equilibrium distribution function for classical system can write in the form [14]:$f_l = Q_l^{-1}\exp\left\{-\int\left(\beta(\mathbf{r})H(\mathbf{r})-\eta(\mathbf{r})n(\mathbf{r})\right)d\mathbf{r}\right\}$ where statistical operator local equilibrium distribution can determine as:

$$Q_l=\int D\Gamma\exp\left\{-\int\left(\beta(\mathbf{r})H(\mathbf{r})-\eta(\mathbf{r})n(\mathbf{r})\right)d\mathbf{r}\right\} \tag{1}$$

The integration in present formula must take over all phase space of system. Must note, that in the case local equilibrium distribution Lagrange multipliers $\beta(\mathbf{r})$ and $\eta(\mathbf{r})$ are function of spatial point. Can introduce the entropy local equilibrium distribution by ratio

$$S=-Sp\left(f_l\ln f_l\right)=\ln Q_l+\int\left(\beta(\mathbf{r})H(\mathbf{r})-\eta(\mathbf{r})n(\mathbf{r})\right)d\mathbf{r} \tag{2}$$

After determination of statistical operator can obtain all thermodynamic parameter nonequilibrium system. For this, can determine thermodynamic relation for the systems. The variation of statistical operator by Lagrange multipliers can write necessary thermodynamic relation in the form:

$$-\frac{\delta\ln Q_l}{\delta\beta(\mathbf{r})}=\langle H(\mathbf{r})\rangle_l-\eta(\mathbf{r})\langle n(\mathbf{r})\rangle_l \tag{3}$$

and

$$\frac{\delta\ln Q_l}{\delta\eta(\mathbf{r})}=\langle n(\mathbf{r})\rangle_l \tag{4}$$

This relation is natural general prolongation, on the case inhomogeneous system, well-known relation which take place in the case equilibrium systems. The conservation number of particles and energy in system can present in form natural relations $\int n(\mathbf{r})d\mathbf{r}=N$ and $\int H(\mathbf{r})d\mathbf{r}=E$.

For further statistical description of nonequilibrium system is necessary determine Hamiltonian of system. In the case of self gravitational system Hamiltonian can present in the form:

$$H(\mathbf{r})=\frac{p^2(\mathbf{r})}{2m}n(\mathbf{r})+\frac{1}{2}\int W(\mathbf{r},\mathbf{r}')n(\mathbf{r})n(\mathbf{r}')d\mathbf{r}' \tag{5}$$

where impulse density $p(\mathbf{r})=\sum_i p_i\delta(\mathbf{r}-\mathbf{r_i})$ and gravitation interaction energy can present in well-known form

$$W(\mathbf{r},\mathbf{r}')=\frac{Gm^2}{|\mathbf{r}-\mathbf{r}'|} \tag{6}$$

G is gravitational constant and m is mass of particle. This Hamiltonian of system is possible to use if take into account consider moving in phase space not compressed gravitational fluid. It is valid for not collision systems and self gravitational system present obvious example of such system.

In the case self gravitational system the nonequilibrium statistical operator can write in the form:

$$Q_l=\int D\Gamma\exp\left\{-\int\left(\beta(\mathbf{r})\frac{p^2(\mathbf{r})}{2m}-\eta(\mathbf{r})\right)n(\mathbf{r})d\mathbf{r}-\frac{1}{2}\int W(\mathbf{r},\mathbf{r}')n(\mathbf{r})n(\mathbf{r}')d\mathbf{r}d\mathbf{r}'\right\} \tag{7}$$

In order to perform a formal integration in second part of this presentation, additional field variables can be introduced making use of the theory of Gaussian integrals [15], [13]:

$$\exp\left\{-\frac{1}{2}\int\beta(\mathbf{r})W(\mathbf{r},\mathbf{r}')n(\mathbf{r})n(\mathbf{r}')d\mathbf{r}d\mathbf{r}'\right\}=\int D\varphi\exp\left\{-\frac{1}{2}\int\beta(\mathbf{r})W^{-1}(\mathbf{r},\mathbf{r}')\varphi(\mathbf{r})\varphi(\mathbf{r}')d\mathbf{r}d\mathbf{r}'-\int\sqrt{\beta(\mathbf{r})}\varphi(\mathbf{r})n(\mathbf{r})\right\} \tag{8}$$

where $D\varphi = \frac{\prod_s d\varphi_s}{\sqrt{\det 2\pi\beta W(\mathbf{r},\mathbf{r}')}}$ and $W^{-1}(\mathbf{r},\mathbf{r}')$ is the inverse operator which satisfies the condition $\int d\mathbf{r}'W^{-1}(\mathbf{r},\mathbf{r}')W(\mathbf{r}',\mathbf{r}'')=\delta(\mathbf{r}-\mathbf{r}'')$. The inverse operator $W^{-1}(\mathbf{r},\mathbf{r}')$ of the gravitational interaction, in continuum limit should be treated in the operator sense [17], i.e.

$$W^{-1}(\mathbf{r},\mathbf{r}')=-\frac{1}{4\pi Gm^2}\triangle_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}') \tag{9}$$

where $\triangle_{\mathbf{r}}$- is Laplace operator in real space. After this manipulation the statistical operator can rewrite in the form:

$$Q_l=\int D\Gamma\int D\varphi\exp\left\{-\int\left(\beta(\mathbf{r})\frac{p^2(\mathbf{r})}{2m}-\eta(\mathbf{r})-\sqrt{\beta(\mathbf{r})}\right)n(\mathbf{r})d\mathbf{r}-\frac{1}{8\pi m^2G}\int\left(\nabla\varphi(\mathbf{r})\right)^2d\mathbf{r}\right\} \tag{10}$$

In this functional integral now can be provide the integration on phase space. The integration over phase space can present as functional integration by $D\Gamma=\frac{Dn(\mathbf{r})Dp(\mathbf{r})}{(2\pi\hbar)^3}$ with regard for the cell volume $(2\pi\hbar)^3$ in the phase space of individual states [12], and take into account the "Pauli principle" for classical system, that own spatial point can not occupier two classical particle. As result can use the relation $\int\exp(-A(\mathbf{r})n(\mathbf{r}))Dn(\mathbf{r})=\frac{1}{\det A(\mathbf{r})}\left\{1-\exp\left(-\int A(\mathbf{r})d\mathbf{r}\right)\right\}$. Now can make functional integration over impulse. If introduce new variable absolute chemical activity $\xi(\mathbf{r})\equiv\exp\eta(\mathbf{r})$ after integration over impulse can write statistical operator in the form:

$$Q_l=\int D\varphi\exp\left\{-\int\left[\frac{1}{8\pi m^2G}\left(\nabla\varphi(\mathbf{r})\right)^2-\ln\left(1-\xi(\mathbf{r})(\frac{2\pi m}{\hbar^2\beta(\mathbf{r})})^{\frac{3}{2}}\exp\sqrt{\beta(r)}\varphi(\mathbf{r})\right)\right]d\mathbf{r}\right\} \tag{11}$$

As shown before [11],[12] in all cases classical statistic $\xi\leq 1$ and can use expansion $\ln(1-\xi A)\approx-\xi A+...$ and rewrite the nonequilibrium statistical operator in more simple form:

$$Q_l=\int D\varphi\exp\left\{\int\left[-\frac{1}{8\pi m^2G}\left(\nabla\varphi(\mathbf{r})\right)^2+\xi(\mathbf{r})\frac{2\pi m}{\hbar^2\beta(\mathbf{r})})^{\frac{3}{2}}\exp\sqrt{\beta(r)}\varphi(\mathbf{r})\right]d\mathbf{r}\right\} \tag{12}$$

In the case constant temperature β and absolute chemical activity ξ the statistical operator fully reconstruct the equilibrium grand canonical partition function [12], [21].

The statistical operator allows obtain use the of efficient methods developed in the quantum field theory without imposing additional restrictions of integration over field variables or the perturbation theory. In our case the nonequilibrium statistical operator can rewrite in the form

$$Q_l=\int D\varphi\exp\left\{-F(\varphi(\mathbf{r}),\xi(\mathbf{r}),\beta(r))\right\} \tag{13}$$

where effective nonequilibrium "free energy" can present as:

$$F(\varphi(\mathbf{r}),\xi(\mathbf{r}),\beta(r))=\int\left[-\frac{1}{8\pi m^2G}\left(\nabla\varphi(\mathbf{r})\right)^2+\xi(\mathbf{r})\frac{2\pi m}{\hbar^2\beta(\mathbf{r})})^{\frac{3}{2}}\exp\sqrt{\beta(r)}\varphi(\mathbf{r})\right]d\mathbf{r} \tag{14}$$

The functional $F(\varphi(\mathbf{r}),\xi(\mathbf{r}),\beta(r))$ depends on distribution of the field variables $\varphi(\mathbf{r}$, chemical activity $\xi(\mathbf{r}$ and temperature $\beta(\mathbf{r})$. The field variable contains the same information as original distribution, information about possible states of the systems. The saddle point method can now be further employed can find the asymptotic value of the statistical operator Q_l; the dominant contribution is given by the states which satisfy the extreme condition for the functional. It's easy to see that saddle point equation present thermodynamic relation and it can write in the other form: as equation for field variable $\frac{\delta F}{\delta\varphi(\mathbf{r})}=0$, the normalization condition $\int d\mathbf{r}\frac{\delta F}{\delta(\eta(\mathbf{r}))}=-\int\frac{\delta F}{\delta(\xi(\mathbf{r}))}\xi(\mathbf{r}))d\mathbf{r}=N$ and low of conservation the energy of system $-\int d\mathbf{r}\frac{\delta F}{\delta(\beta(\mathbf{r}))}\xi(\mathbf{r})=E$. Solution of this equation fully determine all thermodynamic parameter and present general solution for the behaviour of self gravitational system, whether this distribution of particles is spatially inhomogeneous or not. The spatially inhomogeneous solution of this equations corespondent the distribution of interacting particles. Very important note, that only in this approach can take into

account the inhomogeneous distribution of temperature, which can depend from spatial distribution of particle in system. In other approaches the dependence of temperature from spatial point was introduce trough polytrophic dependence temperature from density of particle in equation of state. In present approach this dependence leads from necessary thermodynamic condition and can be determine for different distribution of particles. From normalization condition $\int \rho(\mathbf{r}) d\mathbf{r} = N$ can introduce the density function by definition in the form $\rho(\mathbf{r}) \equiv \xi \frac{2\pi m}{\hbar^2 \beta(\mathbf{r})})^{\frac{3}{2}} \exp(\sqrt{\beta(\mathbf{r})}\varphi(\mathbf{r}))$ and rewrite the necessary equation in more simple presentation. Equation for field variable can write in the form

$$\triangle \varphi(\mathbf{r}) + r_m \sqrt{\beta(\mathbf{r})}\rho(\mathbf{r}) = 0 \tag{15}$$

and equation of conservation energy take the form:

$$\int \frac{\rho(\mathbf{r})}{2\beta(\mathbf{r})}(3 - \sqrt{\beta(r)}\varphi(\mathbf{r}))d\mathbf{r} = E \tag{16}$$

where $r_m \equiv 4\pi G m^2$. In the case absent any interaction, from normalization condition $\int \xi \frac{2\pi m}{\hbar^2 \beta})^{\frac{3}{2}} dV = N$ at constant temperature can obtain the the chemical activity $\xi = \frac{N}{V}\frac{2\pi m}{\hbar^2 \beta})^{-\frac{3}{2}}$ and from equation conservation energy relation between energy and temperature $\frac{3}{2}NT = E$. If use obtained relation can determine usual entropy in the form $S = \ln \frac{N!}{V}\left(\frac{4\pi m E}{3N\hbar^2}\right)^{\frac{3}{2}} + \frac{3N}{2}$ that fully reproduce entropy microcanonical distribution gaseous phase for fixed number of particles and energy in system. In the case homogeneous distribution of particle of self gravitational system can take $\rho \equiv \frac{N}{V} = const$. If temperature T is constant, the equation for field variable φ leads to $\triangle \varphi + \frac{4\pi G m^2}{T}\rho = 0$ and have solution $\varphi = \frac{Gm^2}{T}\frac{1}{r}$. Substitution this solution in the equation for conservation energy $\frac{N}{2V}\int_0^R (3 - \frac{Gm^2}{T}\frac{1}{r})r^2 dr = E$ can obtain for the gravitational gaseous phase in limiting volume of size Rcan obtain that $\frac{3}{2}NT = E + \frac{3Gm^2}{8\pi R}$. This relation completely reproduce well-known relation [6] which was obtained by standard approach.

In general case the distribution of particle in self- gravitational system are inhomogeneous. Inhomogeneous distribution of particle motivate the long-range gravitational interaction. Can consider other simple case. If suppose that exist polynomial relation between density and temperature as $\sqrt{\beta(\mathbf{r})}\rho(\mathbf{r}) = A = const$ and density change in accordance with not divergence of number of particle, that possible if density descend as fourth time of distance from center $\rho(r) = Cr^{-4}$ from normalized condition can obtain, that $C = NR_0$ where R_0 is smallest distance to center where classical particle create close packing structure. From equation by field variable $\varphi(\mathbf{r}) = \frac{r_m A}{r}$ that directly reconstruct behavior of gravitational field. From this condition temperature descend as fourth time of distance and can write that $\sqrt{\beta(\mathbf{r})} = \frac{Ar^4}{C}$ or $T = \frac{C}{A}r^{-8}$. From equation for energy can obtain relation between introducing constants in the form $\frac{C^2}{9AR_0^9} - \frac{ACr_m}{6R_0^6} = E$. In this simple case can obtain all necessary coefficient and spatial dependence density and temperature for inhomogeneous distribution particles of self gravitational system.

Indeed,present nonequilibrium statistical description tell only possible dilute structure in self-gravitational system but not describe meta stable states and tell nothing about time scales a kinetic theory. The statistical operator have not any peculiarity for different value of gravitational field. The problem of description of the self-gravitational system of particles could be solved with current approach which take into account the inhomogeneous distribution of particles and temperature. For the first time turn out well described the formation spatial inhomogeneous distribution particle with accompanied by changing temperature such distribution of interacting particles. More over, the method used can be also applied for further development of physics of self- gravitational and similar systems.

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